ANALYTICAL SOLUTION OF THE BOLTZMANN EQUATION IN A KNUDSEN LAYER

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It is a well-known fact that the determination of the exact boundary conditions for the equations of hydrodynamics entails the solution of the Boltzmann kinetic equation in a Knudsen layer [1-4].

In this layer the distribution function can be sought as the superposition of the Enskog function and a function satisfying the linearized Boltzmann equation [4].

We demonstrate this fact, using the method of matched asymptotic expansions [5].

Consider the boundary-layer flow problem [6]. The dimensionless Boltzmann equation has the form

$$\sqrt{K}\, c_y \frac{\partial f}{\partial y} + K c_x \frac{\partial f}{\partial x} = J(f, f) \tag{1}$$

Here $K = l/L$ is the Knudsen number, $K \ll 1$, l is the mean free path, L is the characteristic scale of the body, c is the molecular velocity, and

$$J(f, f) = \int (f' f_1' - f f_1)\, g b\, db\, d\varepsilon\, dc_1$$

In the Knudsen layer we introduce the following transformation of the normal coordinate y:

$$y_1 = y/\sqrt{K} \tag{2}$$

We then obtain

$$c_y \frac{\partial f}{\partial y_1} + K c_x \frac{\partial f}{\partial x} = J(f, f) \tag{3}$$

Let us assume that at $y_1 = 0$ the function f satisfies the diffusion law of reflection of molecules from the wall

$$f H(c_y) = (n_w/\pi^{3/2}) e^{-c^2} \tag{4}$$

Here $H(c_y)$ is the Heaviside function

$$H(c_y) = 1, \quad c_y > 0; \quad H(c_y) = 0, \quad c_y < 0$$

We seek the solution in both domains as series on $\sqrt{K}$. The outer expansion is

$$F(x, y, \mathbf{c}) = F^{(0)} + \sqrt{K} F^{(1)} + \ldots \tag{5}$$

The inner expansion is

$$f(x, y_1, \mathbf{c}) = f^{(0)} + \sqrt{K} f^{(1)} + \ldots \tag{6}$$

We find the splicing conditions for expansions (5) and (6):

$$f^{(0)}(y_1 \to \infty) = F^{(0)}(x, 0, \mathbf{c}) \tag{7}$$

$$f^{(1)}(y_1 \to \infty) = F^{(1)}(x, 0, \mathbf{c}) + y_1 (\partial F^{(0)}/\partial y) \tag{8}$$

For the function $f^{(0)}$ we deduce the equation

$$c_y (\partial f^{(0)}/\partial y_1) = J(f^{(0)}, f^{(0)}) \tag{9}$$

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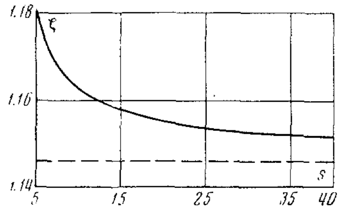

Fig. 1

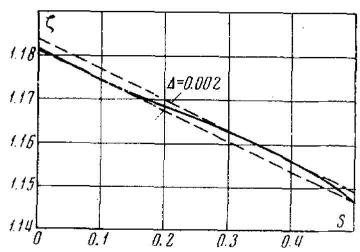

Fig. 2

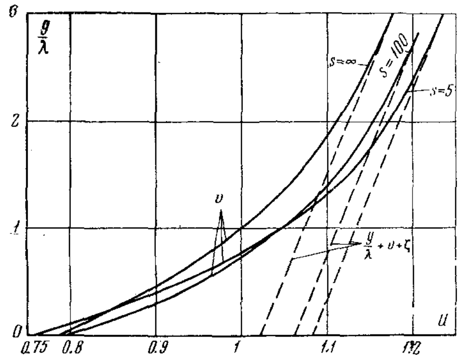

Fig. 3

This equation, subject to the boundary conditions (4) and (7), has a unique solution [7]

$$f^{(0)} = F^{(0)}(x, 0, \mathbf{c}) = (n_w/\pi^{3/2})\, e^{-c^2} \tag{10}$$

In the next-higher approximation for the function $f^{(1)}$ we have

$$c_y(\partial f^{(1)}/\partial y_1) = J(f^{(0)}, f^{(1)}) + J(f^{(1)}, f^{(0)}) \tag{11}$$

We introduce the superposition

$$f^{(1)} = F^{(1)}(x, 0, \mathbf{c}) + y_1(\partial F^{(0)}/\partial y) + f^{(0)}\varphi(x, y_1, \mathbf{c}) \tag{12}$$

We then obtain

$$c_y(\partial\varphi/\partial y_1) = \int f_1^{(0)}(\varphi' + \varphi_1' - \varphi - \varphi_1)\, gb\, db\, d\varepsilon\, d\mathbf{c}_1 \tag{13}$$

$$\varphi(y_1 \to \infty) = 0, \qquad f^{(0)}\varphi(x, 0, \mathbf{c})\, H(c_y) = -F^{(1)}(x, 0, \mathbf{c}) \tag{14}$$

From the second-approximation equation we find for the outer flow that

$$F^{(1)}(x, 0, \mathbf{c}) = f^{(0)}(\varphi_E(0) + n^{(1)}(0) + \mathbf{cu}^{(1)}(0)\, T^{(1)}(c^2 - {}^3/_2)) \tag{15}$$

Here φ_E is the Enskog function [8].

The linearized Boltzmann equation (13) subject to the boundary conditions (14) has a unique solution [9]. A numerical result has been obtained in [10].

We now consider an approximate method for the expansion of the function (for slip flow) in half-space polynomials of the velocities [11-14]

$$\varphi = \varphi^+\left(\frac{1 + \operatorname{sign} c_y}{2}\right) + \varphi^-\left(\frac{1 - \operatorname{sign} c_y}{2}\right), \quad \varphi^\pm = a_0^\pm(y_1)\, c_x + a_1^\pm(y_1)\, c_x c_y \tag{16}$$

After integration of Eq. (13) over the velocities we obtain a set of equations with constant coefficients [14]:

$$\pm\frac{da_0^\pm}{dy_1} + \frac{\sqrt{\pi}}{2}\frac{da_1^\pm}{dy_1} = \pm(a_0^+ - a_0^-)\frac{I_1}{\pi} \pm (a_1^+ + a_1^-)\frac{I_2}{\pi} \tag{17}$$

$$\frac{da_0^\pm}{dy_1}\frac{\sqrt{\pi}}{2} \pm \frac{da_1^\pm}{dy_1} = (a_0^+ - a_0^-)\frac{I_2}{\pi} + (a_1^+ + a_1^-)\frac{I_3}{\pi} \pm (a_1^+ - a_1^-)\frac{I_4}{\pi}$$

The solution of (17) satisfying conditions (14) has the form

$$a_0^{\pm} = b_0^{\pm} e^{-\alpha y_1}, \qquad a_1^{\pm} = b_1^{\pm} e^{-\alpha y_1} \tag{18}$$

The constants $b_0^{\pm}$ and $b_1^{\pm}$ are determined from the boundary conditions, and $\alpha > 0$ is found from the system (17).

At the outer boundary of the Knudsen layer ($y_1 \to \infty$) the solution is continuous.

This result is a consequence of relation (12).

Consequently, the application of the method of half-space moment expansions here is correct (see [11-13]).

The values of the quantities I_1 through I_4 for hard-sphere molecules have been obtained analytically in [4].

The proposed computation technique can be extended to a broader class of potentials.

In particular, for power-law interaction

$$P = \varkappa / r^s \tag{19}$$

it can be shown that

$$I_j = I_j(S), \quad S = \frac{s-5}{2(s-1)}$$

For example,

$$I_2(S) = [c_x \operatorname{sign} c_y, c_x c_y] = -\frac{J_2(S)}{4\sqrt{2}\,\Gamma(1-S)} \frac{\pi}{\lambda}$$
$$\lambda = \frac{2\mu}{\rho V} = \frac{(kT/\varkappa)^{2/(s-1)}}{2^{S+1} n\pi A_2(s)}, \quad A_2(s) = \int_0^\infty (1-\cos^2\chi)\,\rho d\rho \tag{20}$$

Here Γ is the gamma function, and

$$J_2(S) = \int_0^1 \frac{3/2 + (2-t)(6+3S)}{t^S (2-t)^{3/2}} (1-t)^{S+3/2}\, dt$$

The values of I_1, I_2, I_3, and I_4 were determined by numerical integration (for S = 5 to 100). The form of the curve for the slip factor $\zeta(S)$

$$u^{(1)} = \zeta(S)\lambda \frac{\partial u^{(0)}}{\partial y} \tag{21}$$

is shown in Fig. 1. The calculation was carried out with maximum 0.2% error.

The dependence $\zeta(S)$ is shown in Fig. 2. It is evident that within the computational error limits the function $\zeta(S)$ is very nearly linear. The values of $\zeta(S)$ for $S = 0$ and $S = 0.5$ are close to earlier-published results [11, 13, 14].

The velocity profile in the Knudsen layer

$$U = y/\lambda + v(s)e^{-\alpha y} + \zeta(s), \quad u = U\lambda(\partial u/\partial y), \quad s = 5, 100, \infty$$

is illustrated in Fig. 3.

In order to verify the convergence of the method it is advisable that several higher approximations be analyzed for $\varphi^{\pm}$.

The values of $I_1(S)$ through $I_4(S)$ can be used in other linear problems (such as Couette or Poiseuille flow).

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